B.A./B.Sc. 5th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH5DSE12

(Number Theory)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Show that Goldbach conjecture implies that every even integer greater than 5 is a sum of three primes.

(b) For any integers a, b, c prove that $a|b$ and $b|a$ iff $a = \pm b$.

(c) Prove that $(n^2 + 2)$ is not divisible by 4 for any integer n.

(d) Find the remainder when 3^{100} is divided by 5.

(e) State Fermat's Little Theorem.

(f) Show that $19^{20} \equiv 1 \pmod{181}$.

(g) Prove that if $8 \times 7 \equiv 2 \times 7 \pmod 6$ and $(7, 6) = 1$, then $8 \equiv 2 \pmod 6$.

(h) Solve $x^2 + 3x + 11 \equiv 0 \pmod{13}$.

(i) If p is prime, prove that $2(p-3)! + 1 \equiv 0 \pmod p$.

(j) Find the missing digit in the number 287*932 if it is divisible by 13.

(k) Prove that $2^n < n!$ for $n \in \mathbb{N}$ and $n \geq 4$.

(l) If $d_1, d_2, \ldots, d_r$ be the list of all positive divisors of a positive integer n, prove that $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_r} = \frac{\sigma(n)}{n}$.

(m) Solve the linear congruence: $28x \equiv 63 \pmod{105}$.

(n) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ where $p_1, p_2, \ldots, p_r$ are prime to one another, find $\phi(n)$ ($\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers).

(o) Prove that $2^n - 1$ has at least n distinct prime factors.

2. Answer *any four* questions: 5×4=20

(a) (i) Find $\sigma(360)$ and $\sigma(900)$.

(ii) Let $k > 1$ and $2^k - 1$ is a prime. If $n = 2^{k-1}(2^k - 1)$, then show that n is a perfect number. 2+3

(b) Prove that Möbius μ-function is a multiplicative function.

(c) State and prove Euclid's Theorem.

(d) Prove that $an \equiv bn \pmod m$ if and only if $a \equiv b \left(\text{mod } \frac{m}{(m,n)}\right)$, where a, b, m, n are integers.

(e) Show that $3^{4n+2} + 5^{2n+1} \equiv 0$ (mod 14).

(f) Find the primitive roots of 41.

3. Answer *any two* questions: 10×2=20

(a) (i) Prove that every integer $(n > 1)$ can be expressed as a product of finite number of primes.

(ii) Find the remainder when $2^{73} + 14^3$ is divided by 11. 8+2

(b) (i) Find the digit in unit place of 3^{400}.

(ii) State and prove Chinese Remainder Theorem. 2+8

(c) (i) Prove that $7|(2222^{5555} + 5555^{2222})$.

(ii) Solve the linear Diophantine equation: $221x + 35y = 11$ 5+5

(d) (i) Find the least natural number which when divided by 7, 10 and 11 leaves in order the remainders 1, 6 and 2.

(ii) Let p be an odd prime. Then prove that the congruence $x^2 \equiv -1 \pmod p$ has a solution if and only if $p \equiv 1$ (mod 4). 5+5
